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Vibrations of thin piezoelectric shallow shells: Two-dimensional approximation

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Abstract. In this paper we consider the eigenvalue problem for piezoelectric shallow shells and we show that, as the thickness of the shell goes to zero, the eigensolutions of the three-dimensional piezoelectric shells converge to the eigensolutions of a two-dimensional eigenvalue problem.

Keywords. Vibrations; piezoelectricity; shallow shells.

1. Introduction

Lower dimensional models of shells are preferred in numerical computations to three-dimensional models when the thickness of the shells is 'very small'. A lot of work has been done on the lower dimensional approximation of boundary value and eigenvalue problem for elastic plates and shells (cf. [2,3,4,5,6,8,9]). Recently some work has been done on the lower dimensional approximation of boundary value problem for piezoelectric shells (cf. [1]).

In this paper, we would like to study the limiting behaviour of the eigenvalue problems for thin piezoelectric shallow shells. We begin with a brief description of the problem and describe the results obtained.

Let $\hat{\Omega}^\varepsilon = \Phi^\varepsilon(\Omega^\varepsilon), \Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ with $\omega \subset \mathbb{R}^2$, and the mapping $\Phi^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ is given by

$$\Phi^\varepsilon(x^\varepsilon) = (x_1, x_2, \varepsilon\theta(x_1, x_2)) + x_3^\varepsilon a_3^\varepsilon(x_1, x_2)$$

for all $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon$, where θ is an injective mapping of class C^3 and a_3^ε is a unit normal vector to the middle surface $\Phi^\varepsilon(\overline{\omega})$ of the shell. Let $\gamma_0, \gamma_e \subset \partial\omega$ with meas$(\gamma_0) > 0$ and meas$(\gamma_e) > 0$. Let $\hat{\Gamma}_0^\varepsilon = \Phi^\varepsilon(\gamma_0 \times (-\varepsilon, \varepsilon))$ and let $\hat{\Gamma}_e^\varepsilon = \Phi^\varepsilon(\gamma_e \times (-\varepsilon, \varepsilon))$. The shell is clamped along the portion $\hat{\Gamma}_0^\varepsilon$ of the lateral surface.

Then the variational form of the eigenvalue problem consists of finding the displacement vector u^ε, the electric potential φ^ε and $\xi^\varepsilon \in \mathbb{R}$ satisfying eq. (2.21). We then show that the component of the eigenvector involving the electric potential φ^ε can be uniquely determined in terms of the displacement vector u^ε and the problem thus reduces to finding $(u^\varepsilon, \xi^\varepsilon)$ satisfying equations (2.43) and (2.44).

After making appropriate scalings on the data and the unknowns, we transfer the problem to a domain $\Omega = \omega \times (-1, 1)$ which is independent of ε. Then we show that the scaled eigensolutions converge to the solutions of a two-dimensional eigenvalue problem (6.50).

2. The three-dimensional problem

Throughout this paper, Latin indices vary over the set $\{1,2,3\}$ and Greek indices over the set $\{1,2\}$ for the components of vectors and tensors. The summation over repeated indices will be used.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. Let $\gamma_0, \gamma_e \subset \partial\omega$ with meas$(\gamma_0) > 0$ and meas$(\gamma_e) > 0$. Let $\gamma_1 = \partial\omega \backslash \gamma_0$ and $\gamma_s = \partial\omega \backslash \gamma_e$. For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon), \quad \Gamma^{\pm,\varepsilon} = \omega \times \{\pm\varepsilon\}, \quad \Gamma_0^\varepsilon = \gamma_0 \times (-\varepsilon, \varepsilon),$$
$$\Gamma_1^\varepsilon = \gamma_1 \times (-\varepsilon, \varepsilon), \quad \Gamma_e^\varepsilon = \gamma_e \times (-\varepsilon, \varepsilon), \quad \Gamma_s^\varepsilon = \gamma_s \times (-\varepsilon, \varepsilon).$$

Let $x^\varepsilon = (x_1, x_2, x_3^\varepsilon)$ be a generic point on Ω^ε and let $\partial_\alpha = \partial_\alpha^\varepsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\varepsilon = \frac{\partial}{\partial x_3^\varepsilon}$.

We assume that for each ε, we are given a function $\theta^\varepsilon : \omega \to \mathbb{R}$ of class C^3. We then define the map $\phi^\varepsilon : \omega \to \mathbb{R}^3$ by

$$\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) \quad \text{for all } (x_1, x_2) \in \omega. \tag{2.1}$$

At each point of the surface $S^\varepsilon = \phi^\varepsilon(\omega)$, we define the normal vector

$$a^\varepsilon = (|\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1)^{-1/2}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1).$$

For each $\varepsilon > 0$, we define the mapping $\Phi^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ by

$$\Phi^\varepsilon(x^\varepsilon) = \phi^\varepsilon(x_1, x_2) + x_3^\varepsilon a^\varepsilon(x_1, x_2) \quad \text{for all } x^\varepsilon \in \Omega^\varepsilon. \tag{2.2}$$

It can be shown that there exists an $\varepsilon_0 > 0$ such that the mappings $\Phi^\varepsilon : \Omega^\varepsilon \to \Phi^\varepsilon(\Omega^\varepsilon)$ are C^1 diffeomorphisms for all $0 < \varepsilon \le \varepsilon_0$. The set $\hat{\Omega}^\varepsilon = \Phi^\varepsilon(\Omega^\varepsilon)$ is the reference configuration of the shell. For $0 < \varepsilon \le \varepsilon_0$, we define the sets

$$\hat{\Gamma}^{\pm,\varepsilon} = \Phi^\varepsilon(\Gamma^{\pm,\varepsilon}), \quad \hat{\Gamma}_0^\varepsilon = \Phi^\varepsilon(\Gamma_0^\varepsilon), \quad \hat{\Gamma}_1^\varepsilon = \Phi(\Gamma_1^\varepsilon), \quad \hat{\Gamma}_N^\varepsilon = \hat{\Gamma}_i^\varepsilon \cup \hat{\Gamma}^{\pm\varepsilon},$$
$$\hat{\Gamma}_e^\varepsilon = \Phi(\Gamma_e^\varepsilon), \quad \hat{\Gamma}_s^\varepsilon = \Phi(\Gamma_s^\varepsilon), \quad \hat{\Gamma}_{eD}^\varepsilon = \hat{\Gamma}_e^\varepsilon \cup \hat{\Gamma}^{\pm\varepsilon}$$

and we define vectors g_i^ε and $g^{i,\varepsilon}$ by the relations

$$g_i^\varepsilon = \partial_i^\varepsilon \Phi^\varepsilon \quad \text{and} \quad g^{j,\varepsilon} \cdot g_i^\varepsilon = \delta_i^j$$

which form the covariant and contravariant basis respectively of the tangent plane of $\Phi^\varepsilon(\Omega^\varepsilon)$ at $\Phi^\varepsilon(x^\varepsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\varepsilon = g_i^\varepsilon \cdot g_j^\varepsilon \quad \text{and} \quad g^{ij,\varepsilon} = g^{i,\varepsilon} \cdot g^{j,\varepsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\varepsilon} = g^{p,\varepsilon} \cdot \partial_j^\varepsilon g_i^\varepsilon.$$

Note however that when the set Ω^ε is of the special form $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ and the mapping Φ^ε is of the form (2.2), the following relations hold:

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0.$$

The volume element is given by $\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon$ where

$$g^\varepsilon = \det(g^\varepsilon_{ij}).$$

It can be shown that there exist constants g_1 and g_2 such that

$$0 < g_1 \le g^\varepsilon \le g_2 \tag{2.3}$$

for $0 \le \varepsilon \le \varepsilon_0$.

Let $\hat{A}^{ijkl,\varepsilon}, \hat{P}^{ijk,\varepsilon}$ and $\hat{\mathscr{E}}^{ij,\varepsilon}$ be the elastic, piezoelectric and dielectric tensors respectively. We assume that the material of the shell is *homogeneous and isotropic*. Then the elasticity tensor is given by

$$\hat{A}^{ijkl,\varepsilon} = \lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk}), \tag{2.4}$$

where λ and μ are the Lamè constants of the material.

These tensors satisfy the following coercive relations. There exists a constant $C > 0$ such that for all symmetric tensors (M_{ij}) and for any vector $(t_i) \in \mathbb{R}^3$,

$$\hat{A}^{ijkl,\varepsilon}M_{kl}M_{ij} \ge C\sum_{i,j=1}^{3}(M_{ij})^2, \tag{2.5}$$

$$\hat{\mathscr{E}}^{kl,\varepsilon}t_k t_l \ge C\sum_{j=1}^{3}t_j^2. \tag{2.6}$$

Moreover we have the symmetries

$$\hat{A}^{ijkl,\varepsilon} = \hat{A}^{klij,\varepsilon} = \hat{A}^{jikl,\varepsilon}, \quad \hat{\mathscr{E}}^{kl,\varepsilon} = \hat{\mathscr{E}}^{kl,\varepsilon}, \quad \hat{P}^{ijk,\varepsilon} = \hat{P}^{kij,\varepsilon}.$$

Then the eigenvalue problem consists of finding $(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon, \xi^\varepsilon)$ such that

$$\left.\begin{array}{l} -\mathrm{div}\hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) = \xi^\varepsilon\hat{u}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon \\ \hat{\sigma}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu = 0 \text{ on } \hat{\Gamma}^\varepsilon_N \\ \hat{u}^\varepsilon = 0 \text{ on } \hat{\Gamma}^\varepsilon_0 \end{array}\right\}, \tag{2.7}$$

$$\left.\begin{array}{l} \mathrm{div}\hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon) = 0 \text{ in } \hat{\Omega}^\varepsilon \\ \hat{D}^\varepsilon(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon)\nu = 0 \text{ on } \hat{\Gamma}^\varepsilon_s \\ \hat{\varphi}^\varepsilon = 0 \text{ on } \hat{\Gamma}^\varepsilon_{eD}. \end{array}\right\}, \tag{2.8}$$

where

$$\hat{\sigma}^\varepsilon_{ij} = \hat{A}^{ijkl,\varepsilon}\hat{e}^\varepsilon_{ij} - \hat{P}^{kij,\varepsilon}\hat{E}_k, \tag{2.9}$$

$$\hat{D}_k = \hat{P}^{kij,\varepsilon}\hat{e}^\varepsilon_{ij} + \hat{\mathscr{E}}^{kl,\varepsilon}\hat{E}_l, \tag{2.10}$$

where $\hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_i), \hat{\partial}^\varepsilon_i = \partial/\partial\hat{x}^\varepsilon_i$ and $\hat{E}_k(\hat{\varphi}^\varepsilon) = -\bigtriangledown(\hat{\varphi}^\varepsilon)$.

We define the spaces

$$\hat{V}^\varepsilon = \{\hat{v} \in (H^1(\hat{\Omega}^\varepsilon))^3, \hat{v}|_{\hat{\Gamma}^\varepsilon_0} = 0\}, \tag{2.11}$$

$$\hat{\Psi}^\varepsilon = \{\hat{\psi} \in H^1(\hat{\Omega}^\varepsilon), \hat{\psi}|_{\hat{\Gamma}^\varepsilon_{eD}} = 0\}. \tag{2.12}$$

Then the variational form of systems (2.7) and (2.8) is to find $(\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon, \xi^\varepsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon \times \mathbb{R}$ such that

$$\hat{a}^\varepsilon((\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) = \xi^\varepsilon \hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \quad \text{for all } (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) \in \hat{V}^\varepsilon \times \hat{\Psi}^\varepsilon, \tag{2.13}$$

where

$$\begin{aligned}\hat{a}^\varepsilon((\hat{u}^\varepsilon, \hat{\varphi}^\varepsilon), (\hat{v}^\varepsilon, \hat{\psi}^\varepsilon)) = & \int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijkl,\varepsilon} \hat{e}^\varepsilon_{kl}(\hat{u}^\varepsilon) \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) \mathrm{d}\hat{x}^\varepsilon \\ & + \int_{\hat{\Omega}^\varepsilon} \hat{\mathscr{E}}^{ij,\varepsilon} \hat{\partial}^\varepsilon_i \hat{\varphi}^\varepsilon \hat{\partial}^\varepsilon_j \hat{\psi}^\varepsilon \mathrm{d}\hat{x}^\varepsilon \\ & + \int_{\hat{\Omega}^\varepsilon} \hat{P}^{mij,\varepsilon} (\hat{\partial}^\varepsilon_m \hat{\varphi}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{v}^\varepsilon) - \hat{\partial}^\varepsilon_m \hat{\psi}^\varepsilon \hat{e}^\varepsilon_{ij}(\hat{u}^\varepsilon)) \mathrm{d}\hat{x}^\varepsilon,\end{aligned} \tag{2.14}$$

$$\hat{l}^\varepsilon(\hat{v}^\varepsilon, \hat{\psi}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{u}^\varepsilon \cdot \hat{v}^\varepsilon \mathrm{d}\hat{x}^\varepsilon. \tag{2.15}$$

Since the mappings $\Phi^\varepsilon : \overline{\Omega}^\varepsilon \to \overline{\hat{\Omega}}^\varepsilon$ are assumed to be C^1 diffeomorphisms, the correspondences that associate with every element $\hat{v}^\varepsilon \in \hat{V}^\varepsilon$, the vector

$$v^\varepsilon = \hat{v}^\varepsilon \cdot \Phi^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$$

and with every element $\hat{\psi}^\varepsilon \in \hat{\Psi}^\varepsilon$, the function

$$\psi^\varepsilon = \hat{\psi}^\varepsilon \cdot \Phi^\varepsilon : \Omega^\varepsilon \to \mathbb{R}$$

induce bijections between the spaces $\hat{V}^\varepsilon$ and V^ε, and the spaces $\hat{\Psi}^\varepsilon$ and Ψ^ε respectively, where

$$V^\varepsilon = \{v^\varepsilon \in (H^1(\Omega^\varepsilon))^3 | v^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_0\}, \tag{2.16}$$

$$\Psi^\varepsilon = \{\psi^\varepsilon \in H^1(\Omega^\varepsilon) | \psi^\varepsilon = 0 \text{ on } \Gamma^\varepsilon_{eD}\}. \tag{2.17}$$

Then we have

$$\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon(\hat{x}^\varepsilon) = (\partial^\varepsilon_i v^\varepsilon)(g^{i,\varepsilon})_j, \tag{2.18}$$

$$\hat{e}_{ij}(\hat{v})(\hat{x}^\varepsilon) = e^\varepsilon_{k\|l}(v^\varepsilon)(g^{k,\varepsilon})_i (g^{l,\varepsilon})_j, \tag{2.19}$$

where

$$e^\varepsilon_{i\|j}(v^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_i v^\varepsilon_j + \partial^\varepsilon_j v^\varepsilon_i) - \Gamma^{p,\varepsilon}_{ij} v^\varepsilon_p. \tag{2.20}$$

Then the variational form (2.13) posed on the domain Ω^ε is to find $(u^\varepsilon, \varphi^\varepsilon, \xi^\varepsilon) \in V^\varepsilon \times \Psi^\varepsilon \times \mathbb{R}$ such that

$$a^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = \xi^\varepsilon l^\varepsilon(v^\varepsilon, \psi^\varepsilon) \quad \text{for all } (v^\varepsilon, \psi^\varepsilon) \in V^\varepsilon \times \Psi^\varepsilon, \tag{2.21}$$

where

$$\begin{aligned}a^\varepsilon((u^\varepsilon, \varphi^\varepsilon), (v^\varepsilon, \psi^\varepsilon)) = & \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(v^\varepsilon) e^\varepsilon_{i\|j}(v^\varepsilon) \sqrt{g^\varepsilon} \mathrm{d}x^\varepsilon \\ & + \int_{\Omega^\varepsilon} \mathscr{E}^{ij,\varepsilon} \partial^\varepsilon_i \varphi^\varepsilon \partial^\varepsilon_j \psi^\varepsilon \sqrt{g^\varepsilon} \mathrm{d}x^\varepsilon \\ & + \int_{\Omega^\varepsilon} P^{mij,\varepsilon} (\partial^\varepsilon_m \varphi^\varepsilon e^\varepsilon_{i\|j}(v^\varepsilon) \\ & - \partial^\varepsilon_m \psi^\varepsilon e^\varepsilon_{i\|j}(u^\varepsilon)) \sqrt{g^\varepsilon} \mathrm{d}x^\varepsilon,\end{aligned} \tag{2.22}$$

$$l^\varepsilon(v^\varepsilon,\psi^\varepsilon)=\int_{\Omega^\varepsilon} u^\varepsilon\cdot v^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon, \tag{2.23}$$

$$A^{pqrs,\varepsilon}=\hat{A}^{ijkl,\varepsilon}(g^{p,\varepsilon})_i\cdot(g^{q,\varepsilon})_j\cdot(g^{r,\varepsilon})_k\cdot(g^{s,\varepsilon})_l, \tag{2.24}$$

$$\mathscr{E}^{pq,\varepsilon}=\hat{\mathscr{E}}^{ij,\varepsilon}(g^{p,\varepsilon})_i\cdot(g^{q,\varepsilon})_j, \tag{2.25}$$

$$P^{pqr,\varepsilon}=\hat{P}^{ijk,\varepsilon}(g^{p,\varepsilon})_i\cdot(g^{q,\varepsilon})_j\cdot(g^{r,\varepsilon})_k. \tag{2.26}$$

Using the relations (2.3), (2.5) and (2.6), it can be shown that there exists a constant $C>0$ such that for all symmetric tensor (M_{ij}) and for any vector $(t_i)\in\mathbb{R}^3$,

$$A^{ijkl,\varepsilon}M_{kl}M_{ij}\geq C\sum_{i,j=1}^{3}(M_{ij})^2, \tag{2.27}$$

$$\mathscr{E}^{ij,\varepsilon}t_it_j\geq C\sum_{i=1}^{3}t_i^2. \tag{2.28}$$

Clearly the bilinear form associated with the left-hand side of (2.21) is elliptic. Hence by Lax–Milgram theorem, given $f^\varepsilon\in V'^\varepsilon$ and $h^\varepsilon\in\Psi'^\varepsilon$, there exists a unique $(u^\varepsilon,\varphi^\varepsilon)\in V^\varepsilon\times\Psi^\varepsilon$ such that

$$a^\varepsilon((u^\varepsilon,\varphi^\varepsilon),(v^\varepsilon,\psi^\varepsilon))=\langle(f^\varepsilon,h^\varepsilon),(v^\varepsilon,\psi^\varepsilon)\rangle\qquad\forall V^\varepsilon\times\Psi^\varepsilon\in V^\varepsilon\times\Psi^\varepsilon. \tag{2.29}$$

In particular, for each $f^\varepsilon\in(L^2(\Omega^\varepsilon))^3$, there exists a unique solution $(u^\varepsilon,\varphi^\varepsilon)\in V^\varepsilon\times\Psi^\varepsilon$ such that

$$a^\varepsilon((u^\varepsilon,\varphi^\varepsilon),(v^\varepsilon,\psi^\varepsilon))=\int_{\Omega^\varepsilon} f^\varepsilon v^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon\quad\forall v^\varepsilon\times\psi^\varepsilon\in V^\varepsilon\times\Psi^\varepsilon. \tag{2.30}$$

This is equivalent to the following equations.

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon}e^\varepsilon_{k\|l}(u^\varepsilon)e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon+\int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m(\varphi^\varepsilon)e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon$$
$$=\int_{\Omega^\varepsilon} f^\varepsilon v^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon\quad\forall v^\varepsilon\in V^\varepsilon \tag{2.31}$$

and

$$\int_{\Omega^\varepsilon}\mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_i\varphi^\varepsilon\partial^\varepsilon_j\psi^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon=\int_{\Omega^\varepsilon}P^{mij,\varepsilon}\partial^\varepsilon_m\psi^\varepsilon e^\varepsilon_{i\|j}(u^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon\quad\forall\psi^\varepsilon\in\Psi^\varepsilon. \tag{2.32}$$

From relation (2.28), it follows that the bilinear form associated with the left-hand side of (2.32) is Ψ^ε-elliptic.

Also for each $h^\varepsilon\in V^\varepsilon$, the mapping

$$\psi^\varepsilon\to\int_\Omega^\varepsilon P^{mij,\varepsilon}\partial_m\psi^\varepsilon e^\varepsilon_{i\|j}(h^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon$$

defines a linear functional on Ψ^ε. Hence for each $h^\varepsilon\in V^\varepsilon$, there exists a unique $T^\varepsilon(h^\varepsilon)\in\Psi^\varepsilon$ such that

$$\int_{\Omega^\varepsilon}\mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_iT^\varepsilon(h^\varepsilon)\partial^\varepsilon_j\psi^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon=\int_{\Omega^\varepsilon}P^{mij,\varepsilon}\partial^\varepsilon_m\psi^\varepsilon e^\varepsilon_{i\|j}(h^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon\quad\forall\psi^\varepsilon\in\Psi^\varepsilon$$

$$(2.33)$$

and that $T^\varepsilon : V^\varepsilon \to \Psi^\varepsilon$ is continuous.

In particular, it follows from (2.32) and the above equation that $\varphi^\varepsilon = T^\varepsilon(u^\varepsilon)$ and eqs (2.31) and (2.32) become

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(u^\varepsilon) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon + \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m (T^\varepsilon(u^\varepsilon)) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^\varepsilon v^\varepsilon \sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon, \tag{2.34}$$

$$\int_{\Omega^\varepsilon} \mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_i (T^\varepsilon(u^\varepsilon))\partial^\varepsilon_j \psi^\varepsilon \sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m \psi^\varepsilon e^\varepsilon_{i\|j}(u^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \quad \forall \psi^\varepsilon \in \Psi^\varepsilon. \tag{2.35}$$

Lemma 2.1. *For each $h^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, there exists a unique $G^\varepsilon(h^\varepsilon) \in V^\varepsilon$ such that*

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(G^\varepsilon(h^\varepsilon)) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon + \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m (T^\varepsilon(G^\varepsilon(h^\varepsilon))) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} h^\varepsilon v^\varepsilon \sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon \tag{2.36}$$

and that $G^\varepsilon : (L^2(\Omega^\varepsilon))^3 \to V^\varepsilon$ is continuous.

Proof. Let $B^\varepsilon(u^\varepsilon, v^\varepsilon)$ denotes the bilinear form associated with the left-hand side of eq. (2.34). Using (2.35), we have

$$\begin{aligned}
B^\varepsilon(u^\varepsilon, v^\varepsilon) &= \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(u^\varepsilon) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \\
&\quad + \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m (T^\varepsilon(u^\varepsilon)) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \\
&= \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(u^\varepsilon) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \\
&\quad + \int_{\Omega^\varepsilon} \mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_i (T^\varepsilon(u^\varepsilon))\partial^\varepsilon_j (T^\varepsilon(v^\varepsilon))\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(v^\varepsilon) e^\varepsilon_{i\|j}(u^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \\
&\quad + \int_{\Omega^\varepsilon} \mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_i (T^\varepsilon(v^\varepsilon))\partial^\varepsilon_j (T^\varepsilon(u^\varepsilon))\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \\
&= B^\varepsilon(v^\varepsilon, u^\varepsilon).
\end{aligned} \tag{2.37}$$

Also, using (2.35) and the relations (2.27) and (2.28), we have

$$\begin{aligned} B^{\varepsilon}(u^{\varepsilon},u^{\varepsilon}) &= \int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k\|l}(u^{\varepsilon}) e^{\varepsilon}_{i\|j}(u^{\varepsilon}) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &\quad + \int_{\Omega^{\varepsilon}} P^{mij,\varepsilon} \partial^{\varepsilon}_{m}(T^{\varepsilon}(u^{\varepsilon})) e^{\varepsilon}_{i\|j}(u^{\varepsilon}) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &= \int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k\|l}(u^{\varepsilon}) e^{\varepsilon}_{i\|j}(u^{\varepsilon}) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &\quad + \int_{\Omega^{\varepsilon}} \mathscr{E}^{ij,\varepsilon} \partial^{\varepsilon}_{i}(T^{\varepsilon}(u^{\varepsilon})) \partial^{\varepsilon}_{j}(T^{\varepsilon}(u^{\varepsilon})) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &\geq C \|u^{\varepsilon}\|^{2}_{V^{\varepsilon}}. \end{aligned} \tag{2.38}$$

Hence $B^{\varepsilon}(\cdots)$ is symmetric and V^{ε}-elliptic. Hence by Lax–Milgram theorem, there exists a unique $G^{\varepsilon}(h^{\varepsilon})$ satisfying (2.36). Letting $v^{\varepsilon} = G^{\varepsilon}(h^{\varepsilon})$ in (2.36), we get

$$\begin{aligned} &\int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k\|l}(G^{\varepsilon}(h^{\varepsilon})) e^{\varepsilon}_{i\|j}(G^{\varepsilon}(h^{\varepsilon})) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &\quad + \int_{\Omega^{\varepsilon}} P^{mij,\varepsilon} \partial^{\varepsilon}_{m}(T^{\varepsilon}(G^{\varepsilon}(h^{\varepsilon}))) e^{\varepsilon}_{i\|j}(G^{\varepsilon}(h^{\varepsilon})) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &= \int_{\Omega^{\varepsilon}} h^{\varepsilon} G^{\varepsilon}(h^{\varepsilon}) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon}. \end{aligned} \tag{2.39}$$

Using (2.35), it becomes

$$\begin{aligned} &\int_{\Omega^{\varepsilon}} A^{ijkl,\varepsilon} e^{\varepsilon}_{k\|l}(G^{\varepsilon}(h^{\varepsilon})) e^{\varepsilon}_{i\|j}(G^{\varepsilon}(h^{\varepsilon})) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &\quad + \int_{\Omega^{\varepsilon}} \mathscr{E}^{ij,\varepsilon} \partial^{\varepsilon}_{i}(T^{\varepsilon}(G^{\varepsilon}(h^{\varepsilon}))) \partial^{\varepsilon}_{j}(T^{\varepsilon}(G^{\varepsilon}(h^{\varepsilon}))) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon} \\ &= \int_{\Omega^{\varepsilon}} h^{\varepsilon} G^{\varepsilon}(h^{\varepsilon}) \sqrt{g^{\varepsilon}} \mathrm{d}x^{\varepsilon}. \end{aligned} \tag{2.40}$$

Using the relations (2.27) and (2.28), we have

$$\|G^{\varepsilon}(h^{\varepsilon})\|^{2}_{V^{\varepsilon}} \leq C^{\varepsilon} \|G^{\varepsilon}(h^{\varepsilon})\|_{V^{\varepsilon}} \|h^{\varepsilon}\|_{(L^{2}(\Omega^{\varepsilon}))^{3}}. \tag{2.41}$$

Hence

$$\|G^{\varepsilon}(h^{\varepsilon})\|_{V^{\varepsilon}} \leq C^{\varepsilon} \|h^{\varepsilon}\|_{(L^{2}(\Omega^{\varepsilon}))^{3}} \tag{2.42}$$

which implies that G^{ε} is continuous.

It follows from (2.34) and the above lemma that $u^{\varepsilon} = G^{\varepsilon}(f^{\varepsilon})$. Since the inclusion $(H^{1}(\Omega^{\varepsilon}))^{3} \hookrightarrow (L^{2}(\Omega^{\varepsilon}))^{3}$ is compact, it follows that $G^{\varepsilon} : (L^{2}(\Omega^{\varepsilon}))^{3} \to (L^{2}(\Omega^{\varepsilon}))^{3}$ is compact. Also since the bilinear form $B^{\varepsilon}(\cdots)$ is symmetric, it follows that G^{ε} is self-adjoint. Hence from the spectral theory of compact, self-adjoint operators, it follows that there

exists a sequence of eigenpairs $(u^{m,\varepsilon}, \xi^{m,\varepsilon})_{m=1}^{\infty}$ such that

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e^\varepsilon_{k\|l}(u^{m,\varepsilon}) e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon + \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m(T^\varepsilon(u^{m,\varepsilon}))e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \xi^{m,\varepsilon}\int_{\Omega^\varepsilon} u^{m,\varepsilon}v^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \quad \forall v^\varepsilon \in V^\varepsilon, \tag{2.43}$$

$$\int_{\Omega^\varepsilon} \mathscr{E}^{ij,\varepsilon}\partial^\varepsilon_i(T^\varepsilon(u^{m,\varepsilon}))\partial^\varepsilon_j\psi^\varepsilon\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m\psi^\varepsilon e^\varepsilon_{i\|j}(u^{m,\varepsilon})\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon \quad \forall \psi^\varepsilon \in \Psi^\varepsilon, \tag{2.44}$$

$$0 < \xi^{1,\varepsilon} \le \xi^{2,\varepsilon} \le \cdots \le \xi^{m,\varepsilon} \le \cdots \to \infty, \tag{2.45}$$

$$\int_{\Omega^\varepsilon} u^{m,\varepsilon}_i u^{n,\varepsilon}_i\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon = \varepsilon^3\delta_{mn}. \tag{2.46}$$

The sequence $\{u^{m,\varepsilon}\}$ forms a complete orthonormal basis for $(L^2(\Omega))^3$.

Define the Rayleigh quotient $R(\varepsilon)(v^\varepsilon)$ for $v^\varepsilon \in V^\varepsilon$ by

$$R^\varepsilon(v^\varepsilon) = \frac{\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}(v^\varepsilon)e_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon + \int_{\Omega^\varepsilon} P^{mij,\varepsilon}\partial^\varepsilon_m(T^\varepsilon(v^\varepsilon))e^\varepsilon_{i\|j}(v^\varepsilon)\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon}{\int_{\Omega^\varepsilon} v^\varepsilon_i v^\varepsilon_i\sqrt{g^\varepsilon}\mathrm{d}x^\varepsilon}. \tag{2.47}$$

Then

$$\xi^{m,\varepsilon} = \min_{W^\varepsilon \in W^\varepsilon_m}\max_{v^\varepsilon \in W^\varepsilon\setminus\{0\}} R^\varepsilon(v^\varepsilon), \tag{2.48}$$

where W^ε_m denotes the collection of all m-dimensional subspaces of V^ε.

3. The scaled problem

We now perform a change of variable so that the domain no longer depends on ε. With $x = (x_1, x_2, x_3) \in \Omega$, we associate $x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \Omega^\varepsilon$. Let

$$\Gamma_0 = \gamma_0 \times (-1,1), \quad \Gamma_1 = \gamma_1 \times (-1,1), \quad \Gamma^\pm = \omega \times \{\pm 1\},$$
$$\Gamma_e = \gamma_e \times (-1,1), \quad \Gamma_s = \gamma_s \times (-1,1),$$
$$\Gamma_N = \Gamma_1 \cup \Gamma^+ \cup \Gamma^-, \quad \Gamma_{eD} = \Gamma^+ \cup \Gamma^- \cup \Gamma_e.$$

With the functions $\Gamma^{p,\varepsilon}, g^\varepsilon, A^{ijkl,\varepsilon}, P^{ijk,\varepsilon}, \mathscr{E}^{ij,\varepsilon} : \Omega^\varepsilon \to \mathbb{R}$, we associate the functions $\Gamma^p(\varepsilon), g^\varepsilon, A^{ijkl}(\varepsilon), P^{ijk}(\varepsilon), \mathscr{E}^{ij}(\varepsilon) : \Omega \to \mathbb{R}$ defined by

$$\Gamma^p(\varepsilon)(x) := \Gamma^{p,\varepsilon}(x^\varepsilon), \quad g(\varepsilon)(x) = g^\varepsilon(x^\varepsilon), \quad A^{ijkl}(\varepsilon)(x) = A^{ijkl,\varepsilon}(x^\varepsilon), \tag{3.1}$$

$$P^{ijk}(\varepsilon)(x) = P^{ijk,\varepsilon}(x^\varepsilon), \quad \mathscr{E}^{ij}(\varepsilon)(x) = \mathscr{E}^{ij,\varepsilon}(x^\varepsilon). \tag{3.2}$$

Assumption. We assume that the shell is a shallow shell, i.e. there exists a function $\theta \in C^3(\omega)$ such that

$$\phi^\varepsilon(x_1, x_2) = (x_1, x_2, \varepsilon\theta(x_1, x_2)) \quad \text{for all } (x_1, x_2) \in \omega, \tag{3.3}$$

i.e., the curvature of the shell is of the order of the thickness of the shell.

We make the following scalings on the eigensolutions.

$$u_\alpha^{m,\varepsilon}(x^\varepsilon) = \varepsilon^2 u_\alpha^m(\varepsilon)(x), \quad v_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(x), \tag{3.4}$$

$$u_3^{m,\varepsilon}(x^\varepsilon) = \varepsilon u_3^m(\varepsilon)(x), \quad v_3(x^\varepsilon) = \varepsilon v_3(x), \tag{3.5}$$

$$T^\varepsilon(u^{m,\varepsilon}(x^\varepsilon)) = \varepsilon^3 T(\varepsilon)(u^m(\varepsilon)(x)), \quad T^\varepsilon(v(x^\varepsilon)) = \varepsilon^3 T(\varepsilon)(v(x)), \tag{3.6}$$

$$\xi^{m,\varepsilon} = \varepsilon^2 \xi^m(\varepsilon). \tag{3.7}$$

With the tensors $e^\varepsilon_{i\|j}$, we associate the tensors $e_{i\|j}(\varepsilon)$ through the relation

$$e^\varepsilon_{i\|j}(v^\varepsilon)(x^\varepsilon) = \varepsilon^2 e_{i\|j}(\varepsilon; v)(x). \tag{3.8}$$

We define the spaces

$$V(\Omega) = \{v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0\}, \tag{3.9}$$

$$\Psi(\Omega) = \{\psi \in H^1(\Omega), \psi|_{\Gamma_{eD}} = 0\}. \tag{3.10}$$

We denote $\varphi^m(\varepsilon) = T(\varepsilon)(u^m(\varepsilon))$. Then the variational equations (eqs (2.43)–(2.46)) become

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon, u^m(\varepsilon)) e_{i\|j}(\varepsilon, v)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \int_\Omega P^{3kl} \partial_3 \varphi^m(\varepsilon) e_{k\|l}(\varepsilon, v)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega P^{\alpha kl}(\varepsilon) \partial_\alpha \varphi^m(\varepsilon) e_{k\|l}(\varepsilon, v)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \xi^m(\varepsilon) \int_\Omega [\varepsilon^2 u_\alpha^m(\varepsilon) v_\alpha + u_3^m(\varepsilon) v_3]\sqrt{g(\varepsilon)}\mathrm{d}x \quad \text{for all } v \in V(\Omega).
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\int_\Omega \mathscr{E}^{33}(\varepsilon) \partial_3 \varphi^m(\varepsilon) \partial_3 \psi \sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega [\mathscr{E}^{3\alpha}(\varepsilon)(\partial_\alpha \varphi^m(\varepsilon) \partial_3 \psi + \partial_3 \varphi^m(\varepsilon) \partial_\alpha \psi)]\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon^2 \int_\Omega \mathscr{E}^{\alpha\beta}(\varepsilon) \partial_\alpha \varphi^m(\varepsilon) \partial_\beta \psi \sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\varepsilon) \partial_3 \psi e_{k\|l}(\varepsilon, u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon \int_\Omega [P^{\alpha kl}(\varepsilon) \partial_\alpha \psi e_{k\|l}(\varepsilon, u^m(\varepsilon))]\sqrt{g(\varepsilon)}\mathrm{d}x \quad \text{for all } \psi \in \Psi(\Omega),
\end{aligned} \tag{3.12}$$

$$\int_\Omega [\varepsilon^2 u_\alpha^m(\varepsilon) u_\alpha^n(\varepsilon) + u_3^m(\varepsilon) u_3^n(\varepsilon)]\sqrt{g(\varepsilon)}\mathrm{d}x = \delta_{mn}. \tag{3.13}$$

4. Technical preliminaries

The following two lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\varepsilon \to 0$. In the sequel, we denote by $C_1, C_2, \ldots, C_n$ various constants whose values do not depend on ε but may depend on θ.

Lemma 4.1. *The functions $e_{i\|j}(\varepsilon, v)$ defined in* (3.8) *are of the form*

$$e_{\alpha\|\beta}(\varepsilon;v) = \tilde{e}_{\alpha\beta}(v) + \varepsilon^2 e^{\sharp}_{\alpha\|\beta}(\varepsilon;v), \tag{4.1}$$

$$e_{\alpha\|3}(\varepsilon;v) = \frac{1}{\varepsilon}\{\tilde{e}_{\alpha 3}(v) + \varepsilon^2 e^{\sharp}_{\alpha\|3}(\varepsilon;v)\}, \tag{4.2}$$

$$e_{3\|3}(\varepsilon;v) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(v), \tag{4.3}$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3 \partial_{\alpha\beta}\theta, \tag{4.4}$$

$$\tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \tag{4.5}$$

$$\tilde{e}_{33}(v) = \partial_3 v_3 \tag{4.6}$$

and there exists constant C_1 such that

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{\alpha,j} \|e^{\sharp}_{\alpha,j}(\varepsilon;v)\|_{0,\Omega} \le C_1 \|v\|_{1,\Omega} \quad \text{for all } v \in V. \tag{4.7}$$

Also there exist constants C_2, C_3 and C_4 such that

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{x\in\Omega} |g(x) - 1| \le C_2 \varepsilon^2, \tag{4.8}$$

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{x\in\Omega} |A^{ijkl}(\varepsilon) - A^{ijkl}| \le C_3 \varepsilon^2, \tag{4.9}$$

where

$$A^{ijkl} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.10}$$

and

$$A^{ijkl} M_{kl} M_{ij} \ge C_4 M_{ij} M_{ij} \tag{4.11}$$

for $0 < \varepsilon \le \varepsilon_0$ and for all symmetric tensors (M_{ij}).

Proof. The proof is based on Lemma 4.1 of [2].

From relation (2.6) and definition (3.2), it follows that there exists a constant C_5 such that for any vector $(t_i) \in \mathbb{R}^3$,

$$\mathscr{E}^{ij}(\varepsilon) t_i t_j \ge C_5 \sum_{j=1}^{3} t_j^2. \tag{4.12}$$

We assume that there exists functions P^{kij} and $\mathscr{E}^{ij}$ such that

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\Omega} |P^{kij}(\varepsilon) - P^{kij}| \leq C_6\varepsilon, \tag{4.13}$$

$$\sup_{0<\varepsilon\leq\varepsilon_0} \max_{x\in\Omega} |\mathscr{E}^{ij}(\varepsilon) - \mathscr{E}^{ij}| \leq C_7\varepsilon. \tag{4.14}$$

Lemma 4.2. *Let $\theta \in C^3(\omega)$ be a given function and let the functions $\tilde{e}_{ij}$ be defined as in* (4.4)–(4.6). *Then there exists a constant C_8 such that the following generalised Korn's inequality holds*:

$$\|v\|_{1,\Omega} \leq C_8 \left\{ \sum_{i,j} \|\tilde{e}_{ij}(v)\|^2_{0,\Omega} \right\}^{1/2} \tag{4.15}$$

for all $v \in V(\Omega)$ where $V(\Omega)$ is the space defined in (3.9).

Proof. The proof is based on Lemma 4.2 of [2].

5. *A priori* estimates

In this section, we show that for each positive integer m, the scaled eigenvalues $\{\xi^m(\varepsilon)\}$ are bounded uniformly with respect to ε.

Let $\varphi \in H_0^2(\omega)$. Then

$$v_\varphi := (-x_3\partial_1\varphi, -x_3\partial_2\varphi, \varphi) \in V(\Omega) \tag{5.1}$$

and

$$\tilde{e}_{\alpha\beta}(v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \varphi\partial_{\alpha\beta}\theta, \quad \tilde{e}_{i3}(v_\varphi) = 0. \tag{5.2}$$

Hence

$$e_{\alpha\|\beta}(\varepsilon, v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \varphi\partial_{\alpha\beta}\theta + O(\varepsilon^2), \tag{5.3}$$

$$e_{\alpha\|3}(\varepsilon, v_\varphi) = O(\varepsilon), \tag{5.4}$$

$$e_{3\|3}(\varepsilon, v_\varphi) = 0. \tag{5.5}$$

We need the following lemma to prove the boundedness of the scaled eigenvalues.

Lemma 5.1. *There exists a constant $C_9 > 0$ such that*

$$|\partial_3(T(\varepsilon)(v_\varphi))|_{0,\Omega} \leq C_9|\varphi|_{2,\omega}, \tag{5.6}$$

$$|\varepsilon\partial_\alpha(T(\varepsilon)(v_\varphi))|_{0,\Omega} \leq C_9|\varphi|_{2,\omega}. \tag{5.7}$$

Proof. With the scalings (3.3)–(3.7), the variational equation (eq. (2.33)) posed on the domain Ω reads as follows:

For each $h \in (H^1(\Omega))^3$, there exists a unique solution $T(\varepsilon)(h) \in (H^1(\Omega))^3$ such that

$$\begin{aligned}
&\int_\Omega \mathscr{E}^{33}(\varepsilon)\partial_3 T(\varepsilon)(h)\partial_3\psi\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega [\mathscr{E}^{\alpha 3}(\varepsilon)(\partial_\alpha T(\varepsilon)(h)\partial_3\psi + \partial_3 T(\varepsilon)(h)\partial_\alpha\psi)]\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon^2\int_\Omega \mathscr{E}^{\alpha\beta}(\varepsilon)\partial_\alpha T(\varepsilon)(h)\partial_\beta\psi\sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\varepsilon)\partial_3\psi e_{k\|l}(\varepsilon,h)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega P^{\alpha kl}(\varepsilon)\partial_\alpha\psi e_{k\|l}(\varepsilon,h)\sqrt{g(\varepsilon)}\mathrm{d}x \quad \forall\psi\in\Psi.
\end{aligned} \tag{5.8}$$

Taking $h = v_\varphi$ and $\psi = T(\varepsilon)(v_\varphi)$ in the above equation, we have

$$\begin{aligned}
&\int_\Omega \mathscr{E}^{33}(\varepsilon)\partial_3 T(\varepsilon)(v_\varphi)\partial_3 T(\varepsilon)(v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega [\mathscr{E}^{\alpha 3}(\varepsilon)(\partial_\alpha T(\varepsilon)(v_\varphi)\partial_3 T(\varepsilon)(v_\varphi) \\
&\quad + \partial_3 T(\varepsilon)(v_\varphi)\partial_\alpha T(\varepsilon)(v_\varphi))]\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon^2\int_\Omega \mathscr{E}^{\alpha\beta}(\varepsilon)\partial_\alpha T(\varepsilon)(v_\varphi)\partial_\beta T(\varepsilon)(v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\varepsilon)\partial_3 T(\varepsilon)(v_\varphi) e_{k\|l}(\varepsilon,v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega P^{\alpha kl}(\varepsilon)\partial_\alpha T(\varepsilon)(v_\varphi) e_{k\|l}(\varepsilon,v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x.
\end{aligned} \tag{5.9}$$

Using the relations (4.12) and (5.2)–(5.5), it follows that there exists a constant $C_9 > 0$ such that

$$\begin{aligned}
&|\partial_3(T(\varepsilon)(v_\varphi))|^2_{0,\Omega} + |\varepsilon\partial_\alpha(T(\varepsilon)(v_\varphi))|^2_{0,\Omega} \\
&\qquad \le C_9\{|\partial_3 T(\varepsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega} + |\varepsilon\partial_\alpha T(\varepsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega}\}
\end{aligned} \tag{5.10}$$

and hence the result follows.

Theorem 5.2. *For each positive integer m, there exists a constant $C(m) > 0$ such that*

$$\xi^m(\varepsilon) \le C(m). \tag{5.11}$$

Proof. Since problem (3.11) was derived from (2.43) after a change of scale, we still have the variational characterization of the scaled eigenvalues $\xi^m(\varepsilon)$. Let V_m denote the collection of all m-dimensional subspaces of $V(\Omega)$. Then

$$\xi^m(\varepsilon) = \min_{W\in V_m}\max_{v\in W}\frac{N(\varepsilon)(v,v)}{D(\varepsilon)(v,v)}, \tag{5.12}$$

where

$$N(\varepsilon)(v,v) = \int_\Omega A^{ijkl} e_{k\|l}(\varepsilon,v) e_{i\|j}(\varepsilon,v)\sqrt{g(\varepsilon)}\mathrm{d}x + \int_\Omega P^{3kl}\partial_3 T(\varepsilon)(v) e_{k\|l}(\varepsilon,v)\sqrt{g(\varepsilon)}\mathrm{d}x + \varepsilon\int_\Omega P^{\alpha kl}\partial_\alpha T(\varepsilon)(v) e_{k\|l}(\varepsilon,v)\sqrt{g(\varepsilon)}\mathrm{d}x, \tag{5.13}$$

$$D(\varepsilon)(v,v) = \int_\Omega [\varepsilon^2 v_\alpha v_\alpha + v_3 v_3]\sqrt{g(\varepsilon)}\mathrm{d}x. \tag{5.14}$$

Let W_m be the collection of all m-dimensional subspaces of $H_0^2(\omega)$. Let $W \in W_m$. Define

$$\mathbf{W} = \{v_\varphi | \varphi \in W\}. \tag{5.15}$$

It follows that $\mathbf{W} \in V_m$. Hence, it follows from (5.12) that

$$\xi^m(\varepsilon) \le \min_{W\in W_m}\max_{\varphi\in W}\frac{N(\varepsilon)(v_\varphi,v_\varphi)}{D(\varepsilon)(v_\varphi,v_\varphi)}. \tag{5.16}$$

Now,

$$D(\varepsilon)(v_\varphi,v_\varphi) = \int_\Omega [\varepsilon^2 x_3^2|\partial_\alpha\varphi|^2 + |\varphi|^2]\sqrt{g(\varepsilon)}\mathrm{d}x. \geq \int_\omega \varphi^2\mathrm{d}\omega. \tag{5.17}$$

Using the relations (5.3)–(5.5) and Lemma 5.1, it follows that

$$\int_\Omega A^{ijkl} e_{k\|l}(\varepsilon,v_\varphi) e_{i\|j}(\varepsilon,v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \le C\int_\omega |\triangle\varphi|^2\mathrm{d}\omega, \tag{5.18}$$

$$\int_\Omega P^{3kl}\partial_3 T(\varepsilon)(v_\varphi) e_{k\|l}(\varepsilon,v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \le C\int_\omega |\triangle\varphi|^2\mathrm{d}\omega, \tag{5.19}$$

$$\varepsilon\int_\Omega P^{\alpha kl}\partial_\alpha T(\varepsilon)(v_\varphi) e_{k\|l}(\varepsilon,v_\varphi)\sqrt{g(\varepsilon)}\mathrm{d}x \le C\int_\omega |\triangle\varphi|^2\mathrm{d}\omega. \tag{5.20}$$

Hence

$$\xi^m(\varepsilon) \le C\min_{W\in W_m}\max_{\varphi\in W}\frac{\int_\omega|\triangle\varphi|^2\mathrm{d}\omega}{\int_\omega\varphi^2\mathrm{d}\omega} \le C\lambda^m, \tag{5.21}$$

where λ^m is the mth eigenvalue of the two-dimensional elliptic eigenvalue problem

$$\begin{aligned} \triangle^2 u &= \lambda u \quad \text{in } \omega \\ u = \partial_\nu u &= 0 \quad \text{on } \partial\omega. \end{aligned} \tag{5.22}$$

This completes the proof of the theorem on setting $C(m) = C\lambda^m$.

6. The limit problem

Theorem 6.1. (a) *For each positive integer m, there exists* $u^m \in H^1(\Omega), \varphi^m \in L^2(\Omega)$ *and* $\xi^m \in \mathbb{R}$ *such that*

$$u^m(\varepsilon) \to u^m \text{ in } H^1(\Omega), \quad \varphi^m(\varepsilon) \to \varphi^m \text{ in } L^2(\Omega), \tag{6.1}$$

$$(\varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon)) \to (0,0,\partial_3\varphi^m) \text{ in } L^2(\Omega), \tag{6.2}$$

$$\xi^m(\varepsilon) \to \xi^m. \tag{6.3}$$

(b) *Define the spaces*

$$V_H(\omega) = \{(\eta_\alpha) \in (H^1(\omega))^2; \eta_\alpha = 0 \text{ on } \gamma_0\}, \tag{6.4}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}, \tag{6.5}$$

$$V_{KL} = \{v \in H^1(\Omega) | v = \eta_\alpha - x_3\partial_\alpha\eta_3, (\eta_i) \in V_H(\omega) \times V_3(\omega)\}, \tag{6.6}$$

$$\Psi_l = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega)\}, \tag{6.7}$$

$$\Psi_{l0} = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega), \psi|\Gamma^\pm = 0\}. \tag{6.8}$$

Then there exists $(\zeta_\alpha^m, \zeta_3^m) \in V_H \times V_3(\omega)$ *such that*

$$u_\alpha^m = \zeta_\alpha^m - x_3\partial_\alpha\zeta_3^m \quad \text{and} \quad u_3^m = \zeta_3^m, \tag{6.9}$$

$$\varphi^m = (1 - x_3^2)\frac{p^{3\alpha\beta}}{p^{33}}\partial_{\alpha\beta}\xi_3^m \tag{6.10}$$

and $(\zeta^m, \xi^m) \in V_H \times V_3 \times \mathbb{R}$ *satisfies*

$$-\int_\omega m_{\alpha\beta}(\zeta^m)\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n_{\alpha\beta}^\theta(\zeta^m)\partial_{\alpha\beta}\theta\eta_3\mathrm{d}\omega + \frac{2}{3}\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3^m\partial_{\alpha\beta}\eta_3\mathrm{d}\omega$$
$$= \xi^m\int_\omega \zeta_3^m\eta_3\mathrm{d}\omega \quad \forall\eta_3 \in V_3(\omega), \tag{6.11}$$

$$\int_\omega n_{\alpha\beta}^\theta\partial_\beta\eta_\alpha\mathrm{d}\omega = 0 \quad \forall\eta_\alpha \in V_H(\omega), \tag{6.12}$$

where

$$m_{\alpha\beta}(\zeta) = -\left\{\frac{4\lambda\mu}{3(\lambda+4\mu)}\triangle\zeta_3\delta_{\alpha\beta} + \frac{4\mu}{3}\partial_{\alpha\beta}\zeta_3\right\} \tag{6.13}$$

$$n_{\alpha\beta}^\theta(\zeta) = \frac{4\lambda\mu}{\lambda+2\mu}\tilde{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 4\mu\tilde{e}_{\alpha\beta}(\zeta) \tag{6.14}$$

$$p^{33} = \frac{1}{\mu}P^{3\alpha3}P^{3\alpha3} + \frac{1}{\lambda+2\mu}P^{333}P^{333} + \mathscr{E}^{33} \tag{6.15}$$

$$p^{3\alpha\beta} = P^{3\alpha\beta} - \frac{\lambda}{\lambda+2\mu}P^{333}\delta^{\alpha\beta}. \tag{6.16}$$

Proof. For the sake of clarity, the proof is divided into several steps.

Step (i). Define the vector $\tilde{\varphi}_i^m(\varepsilon)$ and the tensor $\tilde{K}^m(\varepsilon) = (\tilde{K}_{ij}^m(\varepsilon))$ by

$$\tilde{\varphi}_i^m(\varepsilon) = (\varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon)), \tag{6.17}$$

$$\tilde{K}_{\alpha\beta}^m(\varepsilon) = \tilde{e}_{\alpha\beta}(u^m(\varepsilon)),\;\; \tilde{K}_{\alpha 3}^m(\varepsilon) = \frac{1}{\varepsilon}\tilde{e}_{\alpha 3}(u^m(\varepsilon)),\;\; \tilde{K}_{33}^m(\varepsilon) = \frac{1}{\varepsilon^2}\tilde{e}_{33}(u^m(\varepsilon)). \tag{6.18}$$

Then there exists a constant $C_{10} > 0$ such that

$$\|u^m(\varepsilon)\|_{1,\Omega} \leq C_{10}, \quad |\tilde{K}_{ij}^m(\varepsilon)|_{0,\Omega} \leq C_{10}, \quad |\tilde{\varphi}_i^m(\varepsilon)|_{0,\Omega} \leq C_{10} \tag{6.19}$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Letting $v = u^m(\varepsilon)$ in (3.11), we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon)(u^m(\varepsilon)) e_{i\|j}(\varepsilon)(u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \int_\Omega P^{3kl}(\varepsilon)\partial_3\varphi^m(\varepsilon) e_{k\|l}(\varepsilon)(u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \varepsilon\int_\Omega P^{\alpha kl}(\varepsilon)\partial_\alpha\varphi^m(\varepsilon) e_{k\|l}(\varepsilon)(u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \xi^m(\varepsilon)\int_\Omega [\varepsilon^2 u_\alpha^m(\varepsilon) u_\alpha^m(\varepsilon) + u_3^m(\varepsilon) u_3^m(\varepsilon)]\sqrt{g(\varepsilon)}\mathrm{d}x.
\end{aligned} \tag{6.20}$$

Letting $\psi = \varphi^m(\varepsilon)$ in (3.12) and using it in the above equation, we get

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon, u^m(\varepsilon)) e_{i\|j}(\varepsilon, u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \int_\Omega \mathscr{E}^{ij}(\varepsilon)\tilde{\varphi}_i^m(\varepsilon)\tilde{\varphi}_j^m(\varepsilon)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&= \xi^m(\varepsilon)\int_\Omega [\varepsilon^2 u_\alpha^m(\varepsilon)\cdot u_\alpha^m(\varepsilon) + u_3^m(\varepsilon) u_3^m(\varepsilon)]\sqrt{g(\varepsilon)}\mathrm{d}x.
\end{aligned} \tag{6.21}$$

Using the coerciveness properties (4.11) and (4.12), the inequality $(a-b)^2 \geq a^2/2 - b^2$ and the generalized Korn's inequality (4.15), we have for $\varepsilon \leq \min\{\varepsilon_0, 1\}$,

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon, u^m(\varepsilon)) e_{i\|j}(\varepsilon, u^m(\varepsilon))\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\quad + \int_\Omega \mathscr{E}^{ij}(\varepsilon)\tilde{\varphi}_i^m(\varepsilon)\tilde{\varphi}_j^m(\varepsilon)\sqrt{g(\varepsilon)}\mathrm{d}x \\
&\geq C_{11}\sum_{i,j}\|e_{i\|j}(\varepsilon, u^m(\varepsilon))\|_{0,\Omega}^2 + C_{11}\sum_i\|\tilde{\varphi}_i^m(\varepsilon)\|_{0,\Omega}^2
\end{aligned}$$

$$
\begin{aligned}
&= C_{11} \sum_{\alpha,\beta} \|\tilde{e}_{\alpha\beta}(u^m(\varepsilon)) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon, u^m(\varepsilon))\|^2_{0,\Omega} \\
&\quad + 2C_{11} \sum_{\alpha} \left\| \frac{1}{\varepsilon} \tilde{e}_{\alpha 3}(u^m(\varepsilon)) + \varepsilon e^\sharp_{\alpha 3}(\varepsilon, u^m(\varepsilon)) \right\|^2_{0,\Omega} \\
&\quad + C_{11} \left\| \frac{1}{\varepsilon^2} \tilde{e}_{33}(u^m(\varepsilon)) \right\|^2_{0,\Omega} + C_{11} \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} |\tilde{K}^m_{ij}(\varepsilon)|^2_{0,\Omega} - C_1^2 (2\varepsilon^2 + \varepsilon^4) \|u^m(\varepsilon)\|^2_{1,\Omega} \right\} \\
&\quad + C_{11} \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} \|\tilde{e}_{ij}(u^m(\varepsilon))\|^2_{0,\Omega} - 3\varepsilon^2 C_1^2 \|u^m(\varepsilon)\|^2_{1,\Omega} \right\} \\
&\quad + C_{11} \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega} \\
&\geq C_{11} \left\{ \frac{1}{2} (C_8)^{-2} - 3\varepsilon^2 C_1^2 \right\} \|u^m(\varepsilon)\|^2_{1,\Omega} + C_{11} \sum_i \|\tilde{\varphi}^m_i(\varepsilon)\|^2_{0,\Omega}. \qquad (6.22)
\end{aligned}
$$

Combining eqs (6.21) and (6.22) with relations (3.13) and (5.11), we get the relation (6.19).

Step (ii). From Step (i) it follows that there exists a subsequence $(\tilde{\varphi}^m_i(\varepsilon))$ and $(\tilde{\varphi}^m_i) \in L^2(\Omega)$ such that

$$
(\varepsilon \partial_1 \varphi^m(\varepsilon), \varepsilon \partial_2 \varphi^m(\varepsilon), \partial_3 \varphi^m(\varepsilon)) \rightharpoonup (\tilde{\varphi}^m_1, \tilde{\varphi}^m_2, \tilde{\varphi}^m_3) \quad \text{in } (L^2(\Omega))^3. \qquad (6.23)
$$

Since Γ_{eD} contains Γ^-, we have

$$
\varphi^m(\varepsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \partial_3 \varphi^m(\varepsilon)(x_1, x_2, s) \mathrm{d}s \qquad (6.24)
$$

and it follows that $\|\varphi^m(\varepsilon)\|_{0,\Omega} \leq \sqrt{2} \|\partial_3 \varphi^m(\varepsilon)\|_{0,\Omega}$. This implies that $\varphi^m(\varepsilon)$ is bounded in $L^2(\Omega)$. Therefore there exists a φ^m in $L^2(\Omega)$ and a subsequence, still indexed by ε, such that $\varphi^m(\varepsilon)$ converges weakly to φ^m. Hence it follows from (6.23) that

$$
(\varepsilon \partial_1 \varphi^m(\varepsilon), \varepsilon \partial_2 \varphi^m(\varepsilon), \partial_3 \varphi^m(\varepsilon)) \rightharpoonup (0, 0, \partial_3 \varphi^m). \qquad (6.25)
$$

Step (iii). From Step (i) it follows that there exists a subsequence, indexed by ε for notational convenience, and functions $u^m \in V(\Omega)$ and $\tilde{K}^m_{ij} \in (L^2(\Omega))^9$ such that

$$
u^m(\varepsilon) \rightharpoonup u^m \quad \text{in } H^1(\Omega), \quad \tilde{K}^m(\varepsilon) \rightharpoonup \tilde{K}^m \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0. \qquad (6.26)
$$

Then there exist functions $(\zeta^m_\alpha) \in H^1(\omega)$ and $\zeta^m_3 \in H^2(\omega)$ satisfying $\zeta^m_i = \partial_\nu \zeta^m_3 = 0$ on γ_0 such that

$$
u^m_\alpha = \zeta^m_\alpha - x_3 \partial_\alpha \zeta^m_3 \quad \text{and} \quad u^m_3 = \zeta^m_3 \qquad (6.27)
$$

and

$$\tilde{K}^m_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u^m), \quad \tilde{K}^m_{\alpha 3} = -\frac{1}{\mu} P^{3\alpha 3}\partial_3 \varphi^m,$$
$$\tilde{K}^m_{33} = -\frac{1}{\lambda + 2\mu}(P^{333}\partial_3\varphi^m + \lambda \tilde{K}^m_{\beta\beta}). \tag{6.28}$$

From definition (6.18) and the boundedness of $(\tilde{K}^m_{ij}(\varepsilon))$, we deduce that

$$\|e_{\alpha 3}(u^m(\varepsilon))\|_{0,\Omega} \le \varepsilon C_{13} \quad \text{and} \quad \|e_{33}(u^m(\varepsilon))\|_{0,\Omega} \le \varepsilon^2 C_{13},$$

where $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. Since norm is a weakly lower semicontinuous function

$$\|e_{i3}(u^m)\|_{0,\Omega} \le \liminf_{\varepsilon\to 0} \|e_{i3}(u^m(\varepsilon))\|_{0,\Omega} = 0, \tag{6.29}$$

we obtain $e_{i3}(u^m) = 0$. Then it is a standard argument that the components u^m_i of the limit u^m are of the form (6.27).

Since $u^m(\varepsilon) \rightharpoonup u^m$ in $H^1(\Omega)$, definition (4.4) of the functions $\tilde{e}_{\alpha\beta}(v)$ shows that the function $\tilde{K}^m_{\alpha\beta}(\varepsilon) = \tilde{e}_{\alpha\beta}(u^m(\varepsilon))$ converges weakly in $L^2(\Omega)$ to the function $\tilde{e}_{\alpha\beta}(u^m)$.

We next note the following result. Let $w \in L^2(\Omega)$ be given; then

$$\int_\Omega w\partial_3 v \mathrm{d}x = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0, \text{ then } w = 0. \tag{6.30}$$

Multiplying (3.11) by ε^2, taking $(v_\alpha) = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega (\lambda \tilde{K}^m_{\sigma\sigma} + (\lambda + 2\mu)\tilde{K}_{33} + P^{333}\partial_3\varphi^m)\partial_3 v_3 \mathrm{d}x = 0 \tag{6.31}$$

which implies $(\lambda \tilde{K}^m_{\sigma\sigma} + (\lambda + 2\mu)\tilde{K}_{33} + P^{333}\partial_3\varphi^m) = 0$ and hence the third relation in (6.28) follows.

Again, multiplying (3.11) by ε, taking $v_3 = 0$ and letting $\varepsilon \to 0$, we get

$$\int_\Omega (\mu \tilde{K}^m_{\alpha 3} + P^{3\alpha 3}\partial_3\varphi^m)\partial_3 v_\alpha \mathrm{d}x = 0 \tag{6.32}$$

which implies $(\mu \tilde{K}^m_{\alpha 3} + P^{3\alpha 3}\partial_3\varphi^m) = 0$ and hence the second relation in (6.28) follows.

Step (iv). The function φ^m is of the form (6.10).

Letting $\varepsilon \to 0$ in eq. (3.12), we get

$$\int_\Omega (P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathscr{E}^{33}\partial_3\varphi^m)\partial_3\psi \mathrm{d}x = 0 \quad \forall \psi \in \Psi(\Omega). \tag{6.33}$$

Since $D(\Omega)$ is dense in Ψ_{l0} (and hence in $\Psi(\Omega)$) for the norm $\|.\|_{\Psi_l}$, eq. (6.33) is equivalent to

$$\partial_3(P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathscr{E}^{33}\partial_3\varphi^m) = 0 \quad \text{in } D'(\Omega) \tag{6.34}$$

which implies that $(P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathscr{E}^{33}\partial_3\varphi^m) = d^1$, with $d^1 \in D(\omega)$. Then

$$\partial_3\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}[\tilde{e}_{\alpha\beta}(\zeta^m) - x_3\partial_{\alpha\beta}\zeta^m_3] - \frac{1}{p^{33}}d^1 \tag{6.35}$$

which gives

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}[x_3\tilde{e}_{\alpha\beta}(\zeta^m) - x_3^2\partial_{\alpha\beta}\zeta_3^m] - \frac{x_3}{p^{33}}d^1 + d^0. \tag{6.36}$$

Since φ^m satisfies the boundary conditions $\varphi^m_{|\Gamma^+} = \varphi^m_{|\Gamma^-} = 0$, we have

$$d^0 = \frac{p^{3\alpha\beta}}{2p^{33}}\partial_{\alpha\beta}\zeta_3^m, \quad d^1 = p^{3\alpha\beta}\tilde{e}_{\alpha\beta}(\zeta^m). \tag{6.37}$$

Thus the conclusion follows.

Step (v). The function (ζ_i^m) satisfies (6.11) and (6.12).

Taking $v \in V_{KL}$ and letting $\varepsilon \to 0$ in (3.11) we get

$$\int_\Omega A^{\alpha\beta kl}\tilde{K}_{kl}^m\tilde{K}_{\alpha\beta}(v)\mathrm{d}x + \int_\Omega P^{3\alpha\beta}\partial_3\varphi^m\tilde{K}_{\alpha\beta}(v)\mathrm{d}x = \xi^m\int_\Omega u_3^m \cdot v_3\mathrm{d}x. \tag{6.38}$$

Replacing u^m and $\tilde{K}_{ij}^m$ by the expressions obtained in (6.27) and (6.28), and taking v of the form

$$v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3 \quad \text{and} \quad v_3 = \eta_3$$

with $(\eta_i) \in V_H(\omega) \times V_3(\omega)$, it is verified that (6.38) coincides with eqs (6.11) and (6.12).

Step (vi). The convergences $u^m(\varepsilon) \rightharpoonup u^m$ in $H^1(\Omega)$ and $\varphi^m(\varepsilon) \rightharpoonup \varphi^m$ in $L^2(\Omega)$ are strong.

To show that the family $(u^m(\varepsilon))$ converges strongly to u^m in $H^1(\Omega)$, by Lemma 4.2, it is enough to show that

$$\tilde{e}_{ij}(u^m(\varepsilon)) \to \tilde{e}_{ij}(u^m) \quad \text{in } L^2(\Omega). \tag{6.39}$$

Since $\tilde{e}_{i3}(u^m) = 0$ and

$$\begin{aligned}&\sum_{i,j}\|\tilde{e}_{ij}(u^m(\varepsilon)) - \tilde{e}_{ij}(u^m)\|_{0,\Omega}^2 \\ &= \sum_{\alpha,\beta}\|\tilde{K}_{\alpha\beta}^m(\varepsilon) - \tilde{K}_{\alpha\beta}^m\|_{0,\Omega}^2 + 2\varepsilon^2\sum_\alpha\|\tilde{K}_{\alpha 3}^m(\varepsilon)\|_{0,\Omega}^2 + \varepsilon^4\|\tilde{K}_{33}^m(\varepsilon)\|_{0,\Omega}^2,\end{aligned} \tag{6.40}$$

convergence (6.39) is equivalent to showing that

$$\tilde{K}^m(\varepsilon) \to \tilde{K}^m \quad \text{in } L^2(\Omega). \tag{6.41}$$

We define a norm on $(L^2(\Omega))^9 \times (L^2(\Omega))^3$ by letting for any matrix $M \in (L^2(\Omega))^9$ and any vector $\chi \in (L^2(\Omega))^3$,

$$\|(M,\chi)\| = \left\{\int_\Omega A^{ijkl}M : M\sqrt{g(\varepsilon)}\mathrm{d}x + \int_\Omega \mathscr{E}^{ij}\chi_i\chi_j\sqrt{g(\varepsilon)}\mathrm{d}x\right\}^{1/2}. \tag{6.42}$$

Let $X^m(\varepsilon)$ be the norm of $(\tilde{K}^m(\varepsilon), \varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon))$ in $(L^2(\Omega))^{12}$. Using the weak convergence equation (eqs (6.25) and (6.26)) and the relation (6.28), it can be shown that

$$\lim_{\varepsilon\to 0}X^m(\varepsilon) = X^m = \left(\int_\Omega A^{ijkl}\tilde{K}^m : \tilde{K}^m\mathrm{d}x + \int_\Omega \mathscr{E}^{33}(\partial_3\varphi^m)^2\mathrm{d}x\right)^{1/2} \tag{6.43}$$

which is the norm of $(\tilde{K}^m, 0, 0, \partial_3\varphi^m)$. Since we have already proved that $(\tilde{K}^m(\varepsilon)$, $\varepsilon\partial_1\varphi^m(\varepsilon)$, $\varepsilon\partial_2\varphi^m(\varepsilon)$, $\partial_3\varphi^m(\varepsilon))$ converges weakly to $(\tilde{K}, 0, 0, \partial_3\varphi^m)$ in $(L^2(\Omega))^{12}$, we have the following strong convergences:

$$\tilde{K}^m(\varepsilon) \to \tilde{K}^m \text{ strongly in } (L^2(\Omega))^9, \tag{6.44}$$

$$(\varepsilon\partial_1\varphi^m(\varepsilon), \varepsilon\partial_2\varphi^m(\varepsilon), \partial_3\varphi^m(\varepsilon)) \to (0, 0, \partial_3\varphi^m) \text{ strongly in } (L^2(\Omega))^3. \tag{6.45}$$

Hence $u^m(\varepsilon)$ converges strongly to u^m in $H^1(\Omega)$ and since $\varphi^m(\varepsilon) - \varphi^m$ is in Ψ_{l0}, the equivalence of norms $\|\psi\|_{\Psi_l}$ and $\psi \to |\partial_3\psi|_\Omega$ in Ψ_{l0} proves that $\varphi^m(\varepsilon)$ converges strongly to φ^m in $L^2(\Omega)$.

Equation (6.12) can be written as

$$\begin{aligned}
&\int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\zeta)\right] \partial_\beta \eta_\alpha \mathrm{d}\omega \\
&= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_\sigma\theta\partial_\sigma\zeta_3)\delta_{\alpha\beta} + \mu(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)\right] \partial_\beta\eta_\alpha \mathrm{d}\omega.
\end{aligned} \tag{6.46}$$

Clearly, the bilinear form

$$\begin{aligned}
\tilde{b}(\zeta_\alpha, \eta_\alpha) &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\zeta)\right] \partial_\beta\eta_\alpha \mathrm{d}\omega \\
&= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_\rho\rho(\zeta) e_{\sigma\sigma}(\eta) + 2\mu e_{\alpha\beta}(\zeta) e_{\alpha\beta}(\eta)\right] \mathrm{d}\omega
\end{aligned} \tag{6.47}$$

is $V_H(\omega)$ elliptic. Also for a given $\zeta_3 \in V_3(\omega)$, the functional

$$\langle \zeta_3, \eta_\alpha \rangle = \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_\sigma\theta\partial_\sigma\zeta_3)\delta_{\alpha\beta} + \mu(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)\right] \partial_\beta\eta_\alpha \mathrm{d}\omega \tag{6.48}$$

is continous on $V_H(\omega)$. Thus, given $\zeta_3 \in V_3(\omega)$, there exists a unique vector $(\zeta_\alpha) \in V_H(\omega)$ such that

$$\tilde{b}(\zeta_\alpha, \eta_\alpha) = \langle \zeta_3, \eta_\alpha \rangle. \tag{6.49}$$

We denote by $T\zeta_3 \in V_H(\omega) \times V_3(\omega)$ the vector (ζ_α, ζ_3). In particular, $T\zeta_3^m = (\zeta_\alpha^m, \zeta_3^m)$.

Substituting this in (6.11), we get

$$b(\zeta_3^m, \eta_3) = \xi^m \int_\omega \zeta^m \eta_3 \mathrm{d}\omega \quad \text{for all } \eta_3 \in V_3(\omega), \tag{6.50}$$

where

$$\begin{aligned}
b(\zeta_3, \eta_3) = &-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n_{\alpha\beta}^\theta(T\zeta_3)\partial_{\alpha\beta}\theta\eta_3 \mathrm{d}\omega \\
&+ \frac{2}{3}\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}} \partial_{\rho\tau}\zeta_3 \partial_{\alpha\beta}\eta_3 \mathrm{d}\omega.
\end{aligned} \tag{6.51}$$

Lemma 6.2. *The bilinear form* $b(\cdots)$ *defined by* (6.51) *is* $V_H(\omega)$*-elliptic and symmetric.*

Proof. It follows from Lemma 6.2 in [8] that the bilinear form $\tilde{b}(\cdots)$ defined by

$$\tilde{b}(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_{\alpha\beta}\theta\eta_3 \mathrm{d}\omega \tag{6.52}$$

is $V_H(\omega)$-elliptic and symmetric. Hence it is clear that $b(\cdots)$ is also $V_H(\omega)$-elliptic and symmetric.

Lemma 6.3. *Let* $(\zeta_3^m, \xi^m), m \geq 1$, *be the eigensolutions of problem* (6.51) *found as limits of the subsequence* $(u^m(\varepsilon), \xi^m(\varepsilon)), m \geq 1$ *of eigensolutions of the problem* (3.11). *Then the sequence* $(\xi^m)_{m=1}^\infty$ *comprises all the eigenvalues, counting multiplicities, of problem* (6.51) *and the associated sequence* $(\zeta_3^m)_{m=1}^\infty$ *of eigenfunctions forms a complete orthonormal set in the space* $V_3(\omega)$.

Proof. The proof is similar to the proof of Lemma 5.4 in [3].

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